$$Cxy-Fx-Gy+H=0\ldots\ldots(7),$$

in which $A=bc-f^2$, $B=ca-g^2$, $C=ab-h^2$, $F=gh-af$, $G=hf-bg$, $H=fg-ch$.

For the conic (4), $A=-1/b^2$, $B=-[(1+\lambda)/a^2]$, $C=[(1-\lambda)/(a^2b^2)]$, $F=-\lambda/a^2b^2$, $G=H=0$, and these in (6) and (7) give

$$(1-\lambda)(x^2-y^2)-2b\lambda y+b^2(1-\lambda)-a^2=0\ldots\ldots(8),$$

$$(1-\lambda)y+b\lambda=0\ldots\ldots(9).$$

Eliminating λ from (8) and (9), the required locus is

$$x^2+y^2+\frac{a^2-2b^2}{b}y=a^2-b^2\quad\ldots\ldots(10).$$

This is a circle, radius $a^2/2b$, or one-half the radius of curvature of (1) at the extremity of the minor axis.

115. Proposed by MARY M. BLAINE, B. Sc., Graduate Student, Drury College, Springfield, Mo.

The locus of a point such that the sum of the squares of its normals form a given ellipsoid is constant, is a co-axial ellipsoid. [From *C. Smith's Solid Analytical Geometry*, page 95.]

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

Let (l, m, n) be the point whose locus is required. Then from the conditions of the problem we have

$$(l-x_1)^2+(l-x_2)^2+(l-x_3)^2+(l-x_4)^2+(l-x_5)^2+(l-x_6)^2$$
$$+(m-y_1)^2+(m-y_2)^2+(m-y_3)^2+(m-y_4)^2+(m-y_5)^2+(m-y_6)^2$$
$$+(n-z_1)^2+(n-z_2)^2+(n-z_3)^2+(n-z_4)^2+(n-z_5)^2+(n-z_6)^2=C^2.$$

This equation can be written more briefly thus :

$$\Sigma(l-x_2)^2+\Sigma(m-y_1)^2+\Sigma(n-z_1)^2=C.$$

$$\therefore\ C(l^2+m^2+n^2)-2[l\Sigma(x_1)+m\Sigma(y_1)+n\Sigma(z_1)]$$
$$+\Sigma(x_1^2)+\Sigma(y_1^2)+\Sigma(z_1^2)=C^2\ldots\ldots(1).$$

The equation of the normal is

$$(a^2/x)(l-x)=(b^2/y)(m-y)=(c^2/z)(n-z)\ldots\ldots(2).$$

Also, $x^2/a^2+y^2/b^2+z^2/c^2=1\ldots\ldots(3)$.

The values of y, z in terms of x from (2) in (3) gives

$$x^2/a^2+\frac{b^2m^2x^2}{(a^2l-a^2x+b^2x)^2}+\frac{c^2n^2x^2}{(a^2l-a^2x+c^2x)^2}=1\ldots\ldots(4).$$

(4) can be written as follows :

$$x^6+Ax^5+Bx^4+Dx^3+Ex^2+Fx+G=0.$$

Then $\Sigma(x_1)=-A$, $\Sigma(x_1x_2)=B$, $\Sigma(x_1^2)=A^2-2B$.

But $A=-\dfrac{2a^2l}{a^2-c^2}-\dfrac{2a^2l}{a^2-b^2}$.

$$B=\frac{a^2(a^2l^2+c^2n^2)}{(a^2-c^2)^2}+\frac{a^2(a^2l^2+b^2m^2)}{(a^2-b^2)^2}+\frac{4a^4l^2}{(a^2-b^2)(a^2-c^2)}-a^2.$$

$$\therefore \Sigma(x_1)=\frac{2a^2l}{a^2-c^2}+\frac{2a^2l}{a^2-b^2}.$$

$$\Sigma(x_1^2)=\frac{2a^2(a^2l^2-c^2n^2)}{(a^2-c^2)^2}+\frac{2a^2(a^2l^2-b^2m^2)}{(a^2-b^2)^2}+2a^2.$$

By symmetry,

$$\Sigma(y_1)=\frac{2b^2m}{b^2-c^2}+\frac{2b^2m}{b^2-a^2}.$$

$$\Sigma(y_1^2)=\frac{2b^2(b^2m^2-c^2n^2)}{(b^2-c^2)^2}+\frac{2b^2(b^2m^2-a^2l^2)}{(b^2-a^2)^2}+2b^2.$$

$$\Sigma(z_1)=\frac{2c^2n}{c^2-b^2}+\frac{2c^2n}{c^2-a^2}.$$

$$\Sigma(z_1^2)=\frac{2c^2(c^2n^2-b^2m^2)}{(c^2-b^2)^2}+\frac{2c^2(c^2n^2-a^2l^2)}{(c^2-a^2)^2}+2c^2.$$

Substituting these values in (1) and reducing we get

$$6(l^2+m^2+n^2)+\frac{2(c^2n^2-a^2l^2)}{a^2-c^2}+\frac{2(b^2m^2-a^2l^2)}{a^2-b^2}+\frac{2(c^2n^2-b^2m^2)}{b^2-c^2}$$
$$=C^2-2(a^2+b^2+c^2).$$

$$\therefore \frac{2(a^4-2a^2b^2-2a^2c^2+3b^2c^2)l^2}{(a^2-c^2)(a^2-b^2)(C^2-2a^2-2b^2-2c^2)}$$

$$+\frac{2(2b^2c^2+2a^2b^2-3a^2c^2-b^4)m^2}{(a^2-b^2)(b^2-c^2)(C^2-2a^2-2b^2-2c^2)}$$

$$+\frac{2(c^4-2a^2c^2-2b^2c^2+3a^2b^2)n^2}{(a^2-c^2)(b^2-c^2)(C^2-2a^2-2b^2-2c^2)}=1.$$

$\therefore\ l^2/R^2+m^2/S^2+n^2/T^2=1$, a co-axial ellipsoid.

CALCULUS.

89. Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, O.

Integrate the equation, $\dfrac{dy}{dx}+y\cos x=\dfrac{\sin 2x}{2}$.

I. Solution by Dr. E. D. ROE, Jr., Norwood, Mass.

Put $\sin x-y=z$; then $\cos xdx-dy=dz$, $dy=\cos xdx-dz$, and the equation becomes

$$\frac{dz}{dx}+(z-1)\cos x=0, \text{ or } \frac{dz}{z-1}+\cos xdx=0.$$

Integrating this, $\log(z-1)+\sin x+\kappa=0$, $z-1=e^{-\sin x-\kappa}$,

$$\text{or } \sin x-y-1=e^{-\sin x-\kappa}=-ce^{-\sin x},\ y=\sin x-1+ce^{-\sin x}.$$

Dr. Roe also furnished a second solution.

II. Solution by F. ANDEREGG, A. M., Professor of Mathematics, Oberlin College, Oberlin, O.

If the equation $\dfrac{dy}{dx}+y\cos x=0$ is solved, the result is $y=C_1e^{-\sin x}$. After substituting in the original equation $\dfrac{dC_1}{dx}$ is found to equal $e^{\sin x}\sin x\cos x$; therefore, $C_1=e^{\sin x}\sin x-e^{\sin x}+C$. And $y=\sin x+Ce^{-\sin x}$.

III. Solution by WALTER H. DRANE, Graduate Student, Harvard University, Cambridge, Mass., and JOHN R. JEFFERY, Student in Ohio State University, Columbus, O.

The general form of this equation is $\dfrac{dy}{dx}+Py=Q$ of which the general integral is $e^{\int Pdx}y=\int e^{\int Pdx}Qdx+C$. Here $\int Pdx=\int\cos xdx=\sin x$.

$$\therefore\ e^{\sin x}y=\int e^{\sin x}\cos x\sin xdx+C.$$

Integrating right member by parts, $e^{\sin x}y=\sin x.e^{\sin x}-e^{\sin x}+c$, or $y=\sin x-1+ce^{-\sin x}$.

[See Johnson's *Differential Equations*, page 35, ex. 7.]